NWL Report No. 1922

AXISYMMETRIC VISCOUS FLUID
MOTIONS AROUND CONICAL SURFACES

by

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Task Assignment
NO. R360FR103/2101/R01101001

May 1964

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ABSTRACT

The introduction of complex Navier-Stokes equations shows that steady axisymmetric motions of viscous incompressible fluids around conical surfaces can be expressed in terms of the corresponding general solution of the Stokes equations of slow motions. The latter integration is accomplished with the aid of slow-motion eigenfunctions with integral eigenvalues for infinite plates and semi-infinite needles and with generally complex eigenvalues for cones and conical corners. The eigenvalues and eigenmotions obtained resemble the corresponding eigenvalues and eigenmotions of the analogous flows past dihedral angles. In particular, the existence of critical and branching eigenvalues reveals that laminar flows past conical surfaces depend on the cone angle in a nonanalytic manner. The investigations include a note on diffusor and jet flows.

FOREWORD

This work was sponsored by the Naval Weapons Laboratory Foundational Research Program. It was performed in the Computation and Analysis Laboratory as part of the project Fluid Dynamics with Applications to the Atmosphere and Oceans (R360FR103/2101/R01101001).

The date of completion was 1 May 1964.

Approved for Release:

/s/ RALPH A. NIEMANN
Acting Technical Director

1. Introduction

Slow motions around the edge of a dihedral angle, which are governed by the linear Stokes equations, have been extensively studied in [4, 6, and 10]. By means of "complex Navier-Stokes" equations introduced in [10] it became possible to reveal four fundamental properties of viscous fluid motions which are important for the general theory of the Navier-Stokes equations:

(I) The solutions of the Navier-Stokes equations are essentially representable by "slow-motion eigenfunctions."

(II) Flow patterns of laminar motions which are "attached" or "separated" at a sharp or blunt edge of an obstacle can be studied in an exact manner.

(III) In contrast to inviscid fluid motions viscous fluid flows around obstacles exhibit "critical body parameters" which prohibit analytic parameter expansions without rigorous justification.

(IV) The Navier-Stokes equations yield solutions which represent "regular motions" with bounded velocities and pressures,and other solutions which describe "singular motions" with infinite pressures and finite or infinite velocities.

The present paper confirms the properties (I) through (IV) by investigating the analogous problem of viscous fluid motions around a conical surface. For the sake of simplicity, the derivations will be confined to axisymmetric flows which depend on two variables. The general problem considered includes as a special case the symmetric flow past a semi-infinite needle which has been briefly studied in [9].

2. Motions Around Conical Surfaces

Consider the steady axisymmetric motion of a viscous incompressible fluid around a conical surface (Figs. 1 and 2), the solid angle of which may be called "cone angle" 2α $(0 \le 2\alpha < 2\pi)$. The complementary angle $2\beta(= 2\pi - 2\alpha)$ is called the "corner angle" of the conical surface. For $2\alpha = 0$ the conical surface degenerates to a semi-infinite thin needle and for $2\alpha = \pi$ to an infinite flat plate. For $0 < 2\alpha < \pi$ the conical surface represents a cone and for $0 < 2\beta < \pi$ a conical corner.

Let (r,θ) denote a polar system in the meridian plane of the axisymmetric motion such that $\theta = \pm\pi$ designates the solid axis of the conical surface (Fig. 1). Furthermore, let (u,v) represent the corresponding radial and tangential velocity components of the motion with the variable pressure p. If r,u,v, and p are reduced to dimensionless variables by appropriate flow parameters which are characteristic for the motions considered (see, for instance, [11]), then the "real Navier-Stokes equations" can be written in the form:

$$\frac{1}{r}(ru)_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}(u_\theta - 2v) - \frac{2}{r^2}(v_\theta + u) = \frac{1}{2}p_r + uu_r + \frac{v}{r}(u_\theta - v)\,, \tag{1}$$

$$\frac{1}{r}(rv)_{rr} + \frac{1}{r^2}v_{\theta\theta} + \frac{1}{r^2}(v\cot\theta)_\theta + \frac{2}{r^2}u_\theta = \frac{1}{2r}p_\theta + uv_r + \frac{v}{r}(v_\theta + u)\,, \tag{2}$$

$$(ur^2\sin\theta)_r + (vr\sin\theta)_\theta = 0. \tag{3}$$

The latter, continuity equation, leads to the introduction of the stream function ψ which is defined by

$$u = -\frac{1}{r^2 \sin\theta}\,\psi_\theta\,, \qquad v = \frac{1}{r\,\sin\theta}\,\psi_r \quad . \tag{4}$$

As was shown in [10] it is advantageous to introduce the concept of "complex laminar flows" which are described by the "complex velocity" (U,V), the "complex pressure" $\tilde{P}$, and the "complex stream function" Ψ, the real parts of which represent "real laminar motions":

$$u = \frac{U+\bar{U}}{2}\,, \quad v = \frac{V+\bar{V}}{2}\,, \quad p = \frac{\tilde{P}+\bar{\tilde{P}}}{2}\,, \quad \psi = \frac{\Psi+\bar{\Psi}}{2} \quad . \tag{5}$$

It is easily verified that complex laminar motions are governed by the following "complex Navier-Stokes equations"

$$\frac{1}{r}\,(rU)_{rr} + \frac{1}{r^2}\,U_{\theta\theta} + \frac{\cot\theta}{r^2}\,(U_\theta - 2V) - \frac{2}{r^2}\,(V_\theta + U) = \tfrac{1}{2}\,\tilde{P}_r$$

$$+ \tfrac{1}{4}\left[(U+\bar{U})U_r + U(U_r+\bar{U}_r) + \frac{V+\bar{V}}{r}\,(U_\theta - V) + \frac{V}{r}\,(U_\theta+\bar{U}_\theta-V-\bar{V})\right], \tag{6}$$

$$\frac{1}{r}\,(rV)_{rr} + \frac{1}{r^2}\,V_{\theta\theta} + \frac{1}{r^2}\,(V\cot\theta)_\theta + \frac{2}{r^2}\,U_\theta = \frac{1}{2r}\,\tilde{P}_\theta$$

$$+ \tfrac{1}{4}\left[(U+\bar{U})V_r + U(V_r+\bar{V}_r) + \frac{V+\bar{V}}{r}\,(V_\theta + U) + \frac{V}{r}\,(V_\theta+\bar{V}_\theta+U+\bar{U})\right], \tag{7}$$

$$(Ur^2\sin\theta)_r + (Vr\,\sin\theta)_\theta = 0, \tag{8}$$

which differ from the real Navier-Stokes equations only in their non-linear inertial terms.

Axisymmetric complex motions around conical surfaces are defined as those solutions of the complex Navier-Stokes equations which satisfy the symmetry condition

$$U(r,-\theta) = + U(r,+\theta), \; V(r,-\theta) = - V(r,+\theta) \tag{9}$$

and the nonslip condition

$$r \geq 0, \quad \theta = \pm \beta : \quad U = 0, \quad V = 0 \quad . \tag{10}$$

When a complex solution is found, the corresponding real motion is obtained by deleting the imaginary parts which have no physical meaning.

With the complex stream function Ψ defined by

$$U = - \frac{1}{r^2 \sin\theta} \Psi_\theta \; , \quad V = \frac{1}{r \sin\theta} \Psi_r \tag{11}$$

the complex Navier-Stokes equations (6) and (7) can be combined to the single equation

$$E^4 \Psi = M(\Psi) \tag{12}$$

for the stream function Ψ. The symbol E^2 denotes the linear differential operator

$$E^2 \Psi = \Psi_{rr} + \frac{1}{r^2} \Psi_{\theta\theta} - \frac{\cot\theta}{r^2} \Psi_\theta \tag{13}$$

and M the nonlinear differential operator

$$M(\Psi) = \frac{\sin\theta}{4} \left[(\Psi+\overline{\Psi})_r \left(\frac{E^2 \Psi}{r^2 \sin^2\theta} \right)_\theta - (\Psi+\overline{\Psi})_\theta \left(\frac{E^2 \Psi}{r^2 \sin^2\theta} \right)_r \right.$$
$$\left. + \Psi_r \left(\frac{E^2\Psi + E^2\overline{\Psi}}{r^2 \sin^2\theta} \right)_\theta - \Psi_\theta \left(\frac{E^2\Psi + E^2\overline{\Psi}}{r^2 \sin^2\theta} \right)_r \right] \quad . \tag{14}$$

For real stream functions ψ the latter nonlinear operator M reduces to the well-known form

$$M(\psi) = \sin\theta\left[\psi_r\left(\frac{E^2\psi}{r^2\sin^2\theta}\right)_\theta - \psi_\theta\left(\frac{E^2\psi}{r^2\sin^2\theta}\right)_r\right] \quad . \tag{15}$$

Since the Navier-Stokes equations are linear in their derivatives of highest order, they may be considered as "almost linear" partial differential equations. Hence, as was pointed out in [10] exact solutions can be constructed in an absolutely linear manner by solving at first the linear homogeneous Stokes equations of slow motions (16), (17), and (18) which neglect all inertial terms of the Navier-Stokes equations. Subsequently, a solution of the complete equations can be found by integrating successively inhomogeneous linear equations which result from the basic slow-motion solution. A detailed description of the latter integration step is presented in [10]. In order to avoid a formal repetition, only the general slow-motion integral may be considered further because it retains the exact properties of the flow around the vertex of the conical surface (see [10]).

3. Separable Slow-motion Integrals

According to the previous section the (complex) Stokes equations of slow motions

$$\frac{1}{r}(rU)_{rr} + \frac{1}{r^2}U_{\theta\theta} + \frac{\cot\theta}{r^2}(U_\theta-2V) - \frac{2}{r^2}(V_\theta+U) = \tfrac{1}{2}\tilde{P}_r \ , \tag{16}$$

$$\frac{1}{r}(rV)_{rr} + \frac{1}{r^2}V_{\theta\theta} + \frac{1}{r^2}(V\cot\theta)_\theta + \frac{2}{r^2}U_\theta = \frac{1}{2r}\tilde{P}_\theta \ , \tag{17}$$

can be combined to the differential equation

$$E^4\Psi = \Psi_{rrrr} + \frac{2}{r^2}(\Psi_{rr\theta\theta} - \cot\theta\Psi_{rr\theta}) - \frac{4}{r^3}(\Psi_{r\theta\theta} - \cot\theta\Psi_{r\theta})$$
$$+ \frac{1}{r^4}\Psi_{\theta\theta\theta\theta} - \frac{2\cot\theta}{r^4}\Psi_{\theta\theta\theta} + \frac{1}{r^4}(8 + 3\cot^2\theta)\Psi_{\theta\theta}$$
$$- \frac{3\cot\theta}{r^4}(3 + \cot^2\theta)\Psi_\theta = 0 \tag{18}$$

for the stream function Ψ. Although the differential equation (18) is not separable in the usual sense, "separable solutions"

$$\Psi = G(r)H(\theta) \tag{19}$$

can still be found in the same generalized manner shown in [10]. A substitution of Eq. (19) into Eq. (18) leads to the equation

$$H'''' - 2\cot\theta H''' + [8 + 3\cot^2\theta + \frac{2}{G}(r^2G'' - 2rG')]H''$$
$$- [3\cot\theta(3 + \cot^2\theta) + \cot\theta\frac{2}{G}(r^2G'' - 2rG')]H' + r^4\frac{G''''}{G}H = 0, \tag{20}$$

which splits into the three ordinary differential equations

$$r^2G'' - 2rG' - \omega_1 G = 0 \quad , \tag{21}$$

$$r^4G'''' - \omega_2 G = 0 \quad , \tag{22}$$

$$H'''' - 2\cot\theta H''' + [8 + 3\cot^2\theta + 2\omega_1]H''$$
$$- [3(3 + \cot^2\theta) + 2\omega_1]\cot\theta H' + \omega_2 H = 0 \quad , \tag{23}$$

where ω_1 and ω_2 are arbitrary complex constants.

The first two equations are of Eulerian type and can be integrated by functions of the form

$$G = r^{\lambda+2} \quad . \tag{24}$$

The corresponding characteristic equations

$$\omega_1 = (\lambda - 1)(\lambda + 2), \quad \omega_2 = (\lambda - 1)\lambda(\lambda + 1)(\lambda + 2) \tag{25}$$

yield the relationship

$$\omega_2 = \lambda(\lambda + 1)\omega_1 \tag{26}$$

between ω_1 and ω_2, so that they both may be replaced by the exponent λ. Since the velocity must be zero at the vertex of the conical surface, the real part of λ must fulfill the "weak regularity condition" (see Eq. (11))

$$\lambda + \bar{\lambda} > 0 \quad . \tag{27}$$

A stronger regularity condition will be required later on the basis of a bounded pressure (see [10]).

The remaining fourth order differential equation (see [1])

$$H'''' - 2\cot\theta H''' + (3\cot^2\theta + 2\lambda^2 + 2\lambda + 4)H'' - (3\cot^2\theta + 2\lambda^2 + 2\lambda + 5)\cot\theta H'$$
$$+ (\lambda - 1)\lambda(\lambda + 1)(\lambda + 2)H = 0 \tag{28}$$

can be factorized in a commutative way:

$$\left[\frac{d^2}{d\theta^2} - \cot\theta \frac{d}{d\theta} + \lambda(\lambda - 1)\right]\left[\frac{d^2}{d\theta^2} - \cot\theta \frac{d}{d\theta} + (\lambda + 1)(\lambda + 2)\right]H = 0. \tag{29}$$

Hence, one has the two second order differential equations ($\lambda \neq -\frac{1}{2}$)

$$H'' - \cot\theta H' + (\lambda + 1)(\lambda + 2)H = 0, \tag{30}$$

$$H'' - \cot\theta H' + (\lambda - 1)\lambda H = 0, \tag{31}$$

which assume the simple forms

$$(1 - t^2)\ddot{H} + (\lambda + 1)(\lambda + 2)H = 0, \tag{32}$$

$$(1 - t^2)\ddot{H} + (\lambda - 1)\lambda H = 0, \tag{33}$$

where the dots denote differentiation with respect to $t = \cos\theta$. The product form (29) of the differential equation (28) is not surprising because the differential equation (30) yields the well-known potential solutions of the same problem which represent simultaneously slow motions.

It is easily verified that the differential equations (32) and (33) are integrable with the aid of the first order associated Legendre functions of first and second kind

$$P^1_\nu(t) = \sqrt{1 - t^2}\,\dot{P}_\nu(t), \quad Q^1_\nu(t) = \sqrt{1 - t^2}\,\dot{Q}_\nu(t) \quad , \tag{34}$$

where the corresponding degrees $\nu = \lambda + 1$ and $\nu = \lambda - 1 \neq 0$ of the Legendre functions $P_\nu(t)$ and $Q_\nu(t)$ may be real or complex. Thus, one arrives at the following set of separable slow-motion integrals of the linear differential equation (18):

$$\Psi_\lambda = r^{\lambda+2}\sin\theta[\tilde{A}_\lambda P^1_{\lambda+1}(\cos\theta) + \tilde{B}_\lambda P^1_{\lambda-1}(\cos\theta) + \tilde{C}_\lambda Q^1_{\lambda+1}(\cos\theta) + \tilde{D}_\lambda Q^1_{\lambda-1}(\cos\theta)] \tag{35}$$

for $\lambda + \bar{\lambda} > 0$ and $\lambda \neq 1$, and

$$\Psi_1 = r^3\sin\theta[\tilde{A}_1 P^1_2(\cos\theta) + \tilde{B}_1 \tan\frac{\theta}{2} + \tilde{C}_1 Q^1_2(\cos\theta) + \tilde{D}_1 \frac{1}{\sin\theta}] \tag{36}$$

for $\lambda = 1$. The coefficients $\tilde{A}_\lambda$, $\tilde{B}_\lambda$, $\tilde{C}_\lambda$, and $\tilde{D}_\lambda$ are arbitrary complex numbers. For vanishing coefficients $\tilde{B}_\lambda$ and $\tilde{D}_\lambda$ the slow motion solutions (35) and (36) represent potential flows.

4. Symmetric Slow-motion Eigenfunctions

The separable slow-motion integrals (35) and (36) represent axi-symmetric slow motions around a conical surface if and only if the symmetry condition (9) and the nonslip condition (10) are satisfied. Unfortunately, the Legendre functions of first and second kind $P_\nu(t)$ and

$Q_\nu(t)$ are generally singular at $t = \pm 1$ ($\theta = 0$ and $\theta = \pi$). Consequently, it is more advantageous to represent the slow-motion integrals (35) by the associated Legendre functions

$$\mathfrak{P}^1_\nu(t) = \sqrt{1 - t^2}\,\dot{\mathfrak{P}}_\nu(t), \quad \mathfrak{Q}^1_\nu(t) = \sqrt{1 - t^2}\,\dot{\mathfrak{Q}}_\nu(t) \tag{37}$$

in which the Legendre function of first kind $\mathfrak{P}_\nu(t)$ is regular at $t = +1$ ($\theta = 0$) for all degrees ν.

Since the Legendre function of second kind $\mathfrak{Q}_\nu(t)$ is singular at $t = +1$ ($\theta = 0$) for all degrees ν, the "axisymmetric slow-motion eigenfunctions" of conical surfaces are of the form

$$\Psi_\lambda = r^{\lambda+2}[A_\lambda H_{\lambda+1}(\cos\theta) + B_\lambda H_{\lambda-1}(\cos\theta)] \tag{38}$$

for $\lambda + \bar{\lambda} > 0$ and $\lambda \neq 1$, and

$$\Psi_1 = r^3[A_1 H_2(\cos\theta) + B_1(1 - \cos\theta)] \tag{39}$$

for $\lambda = 1$. While the coefficients A_λ and B_λ are complex numbers for complex exponents λ, they can be confined to real values for real exponents. The term $H_\nu(t)$ is used as an abbreviation for the function

$$H_\nu(t) = \sqrt{1 - t^2}\,\mathfrak{P}^1_\nu(t) = (1 - t^2)\,\dot{\mathfrak{P}}_\nu(t) \qquad (\nu \neq 0). \tag{40}$$

One derives from Eqs. (32) or (33) the following recurrence formula

$$h_{\nu,n+1} = \frac{n(n - 1) - \nu(\nu + 1)}{2n(n + 1)}\, h_{\nu,n} \quad (n = 1,2,3, \ldots) \tag{41}$$

for the coefficients of the expansion

$$H_\nu(t) = \sum_{n=1}^{\infty} h_{\nu,n}(1 - t)^n. \tag{42}$$

After combining Eqs. (11), (16), (17), (38) and (39) one arrives at the velocity (U_λ, V_λ) and the pressure P_λ (neglecting an arbitrary pressure constant) of the slow motions (38) and (39):

$$U_\lambda = r^\lambda[A_\lambda \dot{H}_{\lambda+1}(\cos\theta) + B_\lambda \dot{H}_{\lambda-1}(\cos\theta)] \quad , \tag{43}$$

$$V_\lambda = \frac{\lambda + 2}{\sin\theta} r^\lambda[A_\lambda H_{\lambda+1}(\cos\theta) + B_\lambda H_{\lambda-1}(\cos\theta)] \, , \tag{44}$$

$$\tilde{P}_\lambda = 4 \, \frac{2\lambda + 2}{\lambda - 1} \, r^{\lambda-1} \dot{H}_{\lambda-1}(\cos\theta) \quad , \tag{45}$$

for $\lambda + \bar{\lambda} > 0$ and $\lambda \neq 1$, and

$$U_1 = r[A_1 \dot{H}_2(\cos\theta) - B_1] \quad , \tag{46}$$

$$V_1 = \frac{3r}{\sin\theta} [A_1 H_2(\cos\theta) + B_1(1 - \cos\theta)] \, , \tag{47}$$

$$\tilde{P}_1 = - 12B_1 \, \log(r \cos^2 \frac{\theta}{2}) \tag{48}$$

for $\lambda = 1$.

The behavior of the pressure functions (45) and (48) shows that the concept of "regular flows" and "singular flows", which was introduced in [10] according as the pressure of the motion is bounded or unbounded, applies also to motions around conical surfaces. In fact, one has the following theorem for conical surfaces which is equivalent to the corresponding theorem for dihedral angles.

THEOREM 1: Eigenfunctions of regular axisymmetric motions around conical surfaces contain only "regular eigenvalues" λ which fulfill the "strong regularity condition"

$$\frac{\lambda + \bar{\lambda}}{2} > 1 \quad . \tag{49}$$

The term "strong" regularity condition is justified in contrast to the notion "weak" regularity condition which was used for Eq. (27). In opposition to other opinions (see, for instance, [5]) it was pointed out in [7, 9,and 10] that only regular motions appear to be physically realistic as they exhibit the features of actually observed flows. Furthermore, in a paper in preparation it will be demonstrated by means of compressible fluids that the acceptance of singular motions leads to such objectionable consequences as, for instance, positive or negative infinite densities.

The nonslip condition (10) and Eqs. (43), (44), (46), and (47) lead to the homogeneous algebraic equations which determine the coefficients A_λ and B_λ of the slow-motion eigenfunctions (38) and (39):

$$\left.\begin{aligned} A_\lambda H_{\lambda+1}(\cos\beta) + B_\lambda H_{\lambda-1}(\cos\beta) &= 0 \\ A_\lambda \dot{H}_{\lambda+1}(\cos\beta) + B_\lambda \dot{H}_{\lambda-1}(\cos\beta) &= 0 \end{aligned}\right\} \tag{50}$$

for $\lambda + \bar{\lambda} > 0$ and $\lambda \neq 1$, and

$$\left.\begin{aligned} A_1 H_2(\cos\beta) + B_1(1 - \cos\beta) &= 0 \\ A_1 \dot{H}_2(\cos\beta) - B_1 \qquad\qquad &= 0 \end{aligned}\right\} \tag{51}$$

for $\lambda = 1$.

With $H_2(t) = (1 - t^2)t$ the algebraic system (51) yields nontrivial solutions A_1 and B_1 if and only if the corner angle $2\beta = 2\beta^*$ satisfies the "cosine condition"

$$\cos^2\beta^* = \cos^2 \frac{\beta^*}{2}, \tag{52}$$

which leads to $\beta^* = 120°$.

THEOREM 2: The "singular eigenvalue" $\lambda = 1$ of axisymmetric motions around conical surfaces exists only for the "limiting cone" with the cone angle $2\alpha^* = 120°$.

The critical angle of the limiting cone may be compared with the critical angle $2\alpha^* \approx 102°$ of the limiting wedge which was found in [10]. Because the limiting wedge angle is smaller than the limiting cone angle, a cone appears to be more critical than a wedge. This is physically plausible since the smaller "volume" of a cone causes less displacement effects in a motion than the larger "volume" of a wedge. As was pointed out in [9 and 10], in a fluid motion the displacement effects caused by the "volumes" of a blunt wedge or cone ($2\alpha^* < 2\alpha$) dominate the friction effects which are caused by the "surfaces" of the wedge or cone. At a sharp wedge or cone ($2\alpha \le 2\alpha^*$) one observes the opposite: the "surface" friction effects exceed the "volume" displacement effects.

The real slow-motion eigenfunction for the limiting cone ($2\alpha = 2\alpha^* = 120°$) with the singular eigenvalue $\lambda = 1$ is (a_1 = real parameter)

$$\psi_1 = a_1 r^3 [\sin\theta \sin 2\theta + \sin^2 \frac{\theta}{2}] \quad . \tag{53}$$

The algebraic system (50) yields nontrivial solutions (A_λ, B_λ) if and only if the Wronskian determinant

$$W(t) = H_{\lambda+1}(t)\dot{H}_{\lambda-1}(t) - \dot{H}_{\lambda+1}(t)H_{\lambda-1}(t) \tag{54}$$

of the two functions $H_{\lambda+1}(t)$ and $H_{\lambda-1}(t)$ vanishes at $t = \cos\beta$. Thus, the condition

$$W(\cos\beta) = H_{\lambda+1}(\cos\beta)\dot{H}_{\lambda-1}(\cos\beta) - \dot{H}_{\lambda+1}(\cos\beta)H_{\lambda-1}(\cos\beta) = 0 \tag{55}$$

determines the "eigenvalues" $\lambda \neq 1$ of axisymmetric slow motions around a conical surface with the corner angle 2β.

After differentiating Eq. (54) and applying Eqs. (32) and (33) one has

$$\dot{W}(t) = H_{\lambda+1}\ddot{H}_{\lambda-1} - \ddot{H}_{\lambda+1}H_{\lambda-1} = 2\,\frac{2\lambda + 1}{1 - t^2}\,H_{\lambda+1}(t)H_{\lambda-1}(t). \tag{56}$$

With $W(1) = 0$ the eigenvalue condition (55) becomes equivalent to the "orthogonality relation"

$$W(\cos\beta) = -\,2(2\lambda + 1)\int_{\cos\beta}^{1} H_{\lambda+1}(t)H_{\lambda-1}(t)\,\frac{dt}{1 - t^2} = 0\ . \tag{57}$$

Thus, the relations (40) lead to the transformed orthogonality condition

$$W(\cos\beta) = -\,2(2\lambda + 1)\int_{\cos\beta}^{1} \mathfrak{P}^{1}_{\lambda+1}(t)\,\mathfrak{P}^{1}_{\lambda-1}(t)dt = 0 \tag{58}$$

for the associated Legendre functions $\mathfrak{P}^{1}_{\lambda+1}(t)$ and $\mathfrak{P}^{1}_{\lambda-1}(t)$.

THEOREM 3: The eigenvalues $\lambda \neq 1$ of axisymmetric slow **motions** around a conical surface with the corner angle β are the arithmetic means of the degrees of those pairs of first order associated Legendre functions of first kind $\mathfrak{P}^{1}_{\lambda+1}(t)$ and $\mathfrak{P}^{1}_{\lambda-1}(t)$ which are orthogonal over the range $\cos\beta \leq t \leq 1$.

It is well known (see Eq. (41)) that the Legendre functions of first kind $\mathfrak{P}_\nu(t)$ (which are regular at $t = +1$ ($\theta = 0$) by definition) assume regular properties at $t = -1$ ($\theta = \pi$) and degenerate to the Legendre polynomials $P_\nu(t)$, if and only if the degree ν is an integer. Furthermore, the first order associated Legendre functions of integral degrees ($\lambda = n$) $\mathfrak{P}^1_{n+1}(t) = P^1_{n+1}(t)$ and $\mathfrak{P}^1_{n-1}(t) = P^1_{n-1}(t)$ are orthogonal over the range ($-1 \leq t \leq +1$). Since their product is an even function of t for all n, they are also orthogonal over the range ($0 \leq t \leq +1$). Hence, the integers $\lambda = n = 2,3,4, \ldots$ are eigenvalues of axisymmetric slow motions normal to an infinite plate ($2\alpha = 2\beta = \pi$) and parallel to a semi-infinite thin needle ($\beta = \pi$). Vice versa, it is not difficult to prove that the integers $\lambda = n = 2,3,4, \ldots$ are the only eigenvalues of axisymmetric slow motions past an infinite plate and around a semi-infinite needle. Indeed, for the infinite plate problem the analyticity of any slow-motion eigenfunction (38) at the origin $r = 0$ follows from the analyticity of the boundary conditions (see [3 and 5]). Hence, the Legendre functions $\mathfrak{P}_{\lambda+1}(t)$ and $\mathfrak{P}_{\lambda-1}(t)$ must degenerate to the Legendre polynomials $P_{n+1}(t)$ and $P_{n-1}(t)$ of the integral degrees $\lambda = n = 2, 3,4, \ldots$. The same follows directly for the semi-infinite needle problem as the eigenfunctions (38) must all vanish at the needle $\theta = \beta = \pi$.

With the eigenvalues of axisymmetric slow motions past an infinite plate $\lambda = n = 2,3,4, \ldots$ one obtains the corresponding real eigenfunctions (a_n = real parameters):

$$\psi_n = a_n r^{n+2} \sin\theta \left[P^1_{n+1}(\cos\theta) - \frac{P^1_{n+1}(0)}{P^1_{n-1}(0)} P^1_{n-1}(\cos\theta) \right] . \tag{60}$$

The eigenvalues of axisymmetric slow motions around a semi-infinite thin needle $\lambda = n = 2,3,4,$ .... lead to the corresponding real eigenfunctions (a_n = real parameters):

$$\psi_n = a_n r^{n+2} \sin\theta \left[P^1_{n+1}(\cos\theta) - \frac{P^1_{n+1}(-1)}{P^1_{n-1}(-1)} P^1_{n-1}(\cos\theta) \right] . \tag{61}$$

This result confirms the integral which was found in [9] for the same needle problem by Taylor series expansions. In particular, one concludes from

$$\begin{aligned} u_n = &- 2a_n r^n \cos\theta \left[\dot{P}_{n+1}(\cos\theta) - \frac{\dot{P}_{n+1}(-1)}{\dot{P}_{n-1}(-1)} \dot{P}_{n-1}(\cos\theta) \right] \\ &+ a_n r^n \sin^2\theta \left[\ddot{P}_{n+1}(\cos\theta) - \frac{\dot{P}_{n+1}(-1)}{\dot{P}_{n-1}(-1)} \ddot{P}_{n-1}(\cos\theta) \right] \end{aligned} \tag{62}$$

that all radial velocity components u_n vanish at $\theta = 0$. Hence, as was pointed out in [9] any viscous fluid passes along an infinitely thin needle (which has neither volume causing displacement nor surface causing friction) in an undisturbed manner without producing any shear stress at the needle.

Since the first order associated Legendre functions of first kind and integral degree $P^1_{n+1}(t)$ and $P^1_{n-1}(t)$ are only orthogonal over the ranges $(0 \le t \le 1)$ and $(-1 \le t \le +1)$, one has the following theorem.

THEOREM 4: Regular integral eigenvalues $\lambda(\neq 1)$ of axisymmetric slow motions around a conical surface exist if and only if the conical surface degenerates to an infinite plate ($2\alpha = 2\beta = \pi$) or a semi-infinite needle ($2\alpha = 0$) in which cases all eigenvalues coincide with the integers $\lambda = n = 2,3,4, \ldots$.

For axisymmetric slow motions around cones ($0 < 2\alpha < \pi$) and into conical corners ($0 < 2\beta < \pi$) the leading eigenvalues $\lambda \neq 1$ have been computed by solving Eq. (55). This has been accomplished by truncating the series expansion of $W(\cos\beta)$ after some appropriate term of N-th order in $(1 - \cos\beta)$. With the recurrence formula (41) the remaining polynomial of order $(2N - 3)$ in λ

$$W(\cos\beta) \approx - \sum_{n=2}^{N} \sum_{j=1}^{n} (1 - \cos\beta)^{n} (n - 2j + 1) h_{\lambda+1,j} h_{\lambda-1,n-j+1} = 0 \qquad (63)$$

retained sufficient accuracy for the leading eigenvalues of interest. Numerical results are presented in Tables 1 and 2 and in Fig. 3.

The numerical results confirm theorem 4 and show that almost all of the slow-motion eigenfunctions of cones ($0 < 2\alpha < \pi$) and of conical corners ($0 < 2\beta < \pi$) are characterized by complex eigenvalues. A comparison with the results presented in [10] reveals that the eigenvalues of symmetric slow motions past dihedral angles and around conical surfaces display essentially the same involved structure. In particular, the existence of branch points shows that the eigenvalues of slow motions around dihedral angles and conical surfaces depend on the corresponding wedge or cone angles in a nonanalytic manner.

It is easily verified that the real eigenfunctions of slow motions past conical surfaces, which are characterized by complex eigenvalues, describe eddy-type flows of the same sort as the analogous eigenfunctions of slow motions around dihedral angles (see [4, 6, and 10]). However, the general real solution, which is the real part of the linear combination (see [2, 3, 5, and 11])

$$\Psi = \Sigma\Psi_\lambda \tag{64}$$

where the sum must be extended over all eigenvalues λ with non-negative imaginary parts, includes also the eddy-free motions of practical interest.

5. Note on Diffusor and Jet Flows

Since the product (29) is commutative, the complex exponents λ are symmetric to the point $\lambda = -1/2$. Hence if λ is an eigenvalue of a conical surface, which satisfies the strong regularity condition $\lambda + \bar{\lambda} > 2$ for the vertex $r = 0$, then $\mu = -(\lambda + 1)$ is an eigenvalue of the same surface which fulfills the strong regularity condition $\mu + \bar{\mu} < -4$ at infinity ($r = \infty$), and vice versa. The corresponding eigenfunction $\Psi_\mu(r,\theta) = r^3 \Psi_\lambda(\frac{1}{r},\theta)$ represents a "regular" slow motion which has neither sources nor sinks at infinity.

If one applies the same "inversion procedure" (see [10]) to a slow-motion eigenfunction of a conical surface which is weakly singular at the vertex $r = 0$ ($0 < \lambda + \bar{\lambda} \leq 2$), then one arrives at a slow-motion eigenfunction which describes a source or sink flow at infinity ($r = \infty$). This reveals the fact that weakly singular motions around sharp cones with angles $2\alpha \leq 2\alpha^* = 120°$ (see Theorem 2) are truly "singular" as they exhibit a source or sink of infinitesimal strength at the vertex of the cone (see [9 and 10]).

The reversible inversion property of regular motions past conical surfaces is only unilateral for weakly singular motions. Indeed, while the singular eigenvalue $\lambda = 1$ exists only for the limiting cone with the angle $2\alpha^* = 120°$, the eigenvalue $\mu = -(\lambda + 1) = -2$ exists for all conical surfaces ($0 \leq 2\alpha < 2\pi$). The corresponding slow-motion eigenfunction is (a_{-2} = real parameter)

$$\psi_{-2} = 2a_{-2}(\cos\theta - 1)[(\cos\theta + 1)\cos\theta - (3\cos^2\beta - 1)], \tag{65}$$

which differs from Ackerberg's solution presented in [1] by an additive constant that adjusts the stream function $\psi_{-2}(r,\theta)$ to the conventional condition $\psi_{-2}(r,0) \equiv 0$. It may be noted that the slow-motion eigenfunction (65) excludes the possibility of Ackerberg's non-existence paradox for the infinite flat plate ($2\alpha = 2\beta = \pi$). In fact, the general solution Ψ of the complex Navier-Stokes equation (12), which is bounded at $r = \infty$, is of the form (see [10])

$$\Psi = \Psi_S(r,\theta) + \Psi_M(r,\theta) \tag{66}$$

where Ψ_S is the general solution of the analogous slow-motion problem containing the r-independent eigenfunction (65) and all eigenfunctions that are regular at $r = \infty$. Accordingly, Ackerberg's solution (see [1]) is in error because the function $\Psi_M(r,\theta)$, which adjusts Ψ_S to a solution of the full Navier-Stokes equation (12), cannot degenerate to a finite sum, unless it vanishes identically together with $\Psi_S \equiv 0$ (see [10]). Furthermore, Ackerberg based his solution only on the slow-motion solution (65) and failed to observe the existence of an infinite number of slow-motion eigenfunctions which are all regular at $r = \infty$. As was pointed out in [10] for $0 < 2\alpha < \pi$ and $0 < 2\beta < \pi$ the function Ψ_M

contains only terms of the separable form (38) because the eigenvalues $\mu = -(\lambda + 1)$ constitute an arithmetic progression only for $2\alpha = 0$ and $2\alpha = 2\beta = \pi$ (see Theorem 4). In the latter case the fundamental system of separable slow-motion solutions (38) must be extended by the so-called "associated" separable slow-motion solutions of order m

$$\Psi_\lambda^m = \frac{\partial^m}{\partial\lambda^m}\Psi_\lambda \qquad (m = 1, 2, 3, \dots), \tag{67}$$

which were introduced in [10].

For example, the associated separable solution of first order is of the form (see Eq. (38)):

$$\begin{aligned} \Psi_\lambda^1 &= r^{\lambda+2}\log r[A_\lambda^1 H_{\lambda+1}(\cos\theta) + B_\lambda^1 H_{\lambda-1}(\cos\theta)] \\ &+ r^{\lambda+2}[A_\lambda^1 \frac{\partial}{\partial\lambda} H_{\lambda+1}(\cos\theta) + B_\lambda^1 \frac{\partial}{\partial\lambda} H_{\lambda-1}(\cos\theta)]. \end{aligned} \tag{68}$$

In general, the solutions (67) involve the functions $(\log r)^m$ $(m = 0, 1, 2, \dots)$ and the "associated functions"

$$H_\nu^m = \frac{\partial^m}{\partial\nu^m} H_\nu(\cos\theta) \qquad (m = 1, 2, 3, \dots) \quad . \tag{69}$$

The latter functions satisfy the recurrence relation $(m = 1, 2, 3, \dots)$

$$(1-t^2)\ddot{H}_\nu^m + \nu(\nu+1)H_\nu^m + m(2\nu+1)H_\nu^{m-1} + m(m-1)H_\nu^{m-2} = 0 . \tag{70}$$

For instance, the first order associated function $H_\nu^1(t)$ is a particular solution of the inhomogeneous differential equation

$$(1-t^2)\ddot{H}_\nu^1 + \nu(\nu+1)H_\nu^1 = -(2\nu+1)H_\nu \quad , \tag{71}$$

which is regular at $t = 1$. Since the inhomogeneous portion of Eq. (71) is a homogeneous solution of the reduced equation, no solution H_ν^1 degenerates to a polynomial even if the exponent ν is an integer. Nevertheless, an exact solution can be found in the form

$$H_\nu^1 = S_\nu^1(t) + \frac{\nu}{|\nu|} H_\nu(t) \log(1+t) \qquad (\nu \neq 0, -1), \tag{72}$$

in which S_ν^1 reduces to a polynomial of order $n = (|\nu + 1/2| - 1/2)$ if the exponent ν is an integer $\nu = 1, \pm 2, \pm 3, \ldots$. After substituting Eq. (72) into Eq. (71) and applying Eq. (40) one arrives at the differential equation ($\nu \neq 0, - 1$)

$$(1-t^2)\ddot{S}_\nu^1 + \nu(\nu+1)S_\nu^1$$

$$= 2\frac{\nu}{|\nu|}(1-t)\left[\nu(\nu+1)\mathfrak{P}_\nu - (|\nu+\tfrac{1}{2}|+\tfrac{1}{2})t\dot{\mathfrak{P}}_\nu - (|\nu+\tfrac{1}{2}|-\tfrac{1}{2})\dot{\mathfrak{P}}_\nu\right], \tag{73}$$

in which for $\nu = 1, \pm 2, \pm 3, \ldots$ the inhomogeneous portion reduces to a polynomial of order $n = (|\nu + 1/2| - 1/2)$ which is lower than the order of the homogeneous polynomial solution $H_\nu(t)$. Hence, the differential equation (73) yields, indeed, a polynomial solution of the order stated above.

According to these derivations the general solution of the full Navier-Stokes equations, which describes laminar motions along an infinite flat plate ($2\alpha = 2\beta = \pi$) with an orifice at the origin $r = 0$ and with a source or sink of finite strength at infinity ($r = \infty$), is of the form (see [10]):

$$\psi = \sum_{n=2}^{\infty} \phi_{-n}(r,\theta), \quad \phi_{-n} = r^{2-n} \sum_{j=0}^{n-2} F_j(\theta)(\log r)^j . \tag{74}$$

After some algebra one arrives at the following two leading terms of the asymptotic expansion (74) (a_{-2} and a_{-3} are real parameters):

$$\phi_{-2} = \psi_{-2} = 2a_{-2}(t^3 - 1) \qquad (t = \cos\theta), \tag{75}$$

$$\phi_{-3} = \frac{t^2}{r}(1-t)[(1+t)(a_{-3}+9a_{-2}^2 \log \frac{1+t}{r}) - a_{-2}^2(3+2t+2t^2)] . \tag{76}$$

It may be noted that the analogous needle ($2\alpha = 0$) and limiting cone ($2\alpha = 2\alpha^* = 120°$) problems, which also seem to have caused certain difficulties in Ackerberg's investigations [1], are solvable without the knowledge of the associated separable slow-motion solutions (67).

Since the slow-motion solution (65) is the leading term in any asymptotic expansion of a laminar flow along a conical surface with an orifice at the vertex $r = 0$ and a source and sink distribution of finite strength at $r = \infty$, it is useful to investigate the corresponding flow pattern. As the velocity components

$$u_{-2} = 6a_{-2} \frac{1}{r^2} (\cos^2\theta - \cos^2\beta), \quad v_{-2} \equiv 0 \tag{77}$$

show, one has a strictly radial diffusor-type flow for all corner angles $0 < 2\beta \leq \pi$ (see Fig. 4a). However, for all cone angles $0 < 2\alpha < \pi$ the motion assumes the character of a jet-type flow as shown in Fig. 4b. In the limiting case of the semi-infinite needle $2\alpha = 0$ the motion is again of the diffusor type. A comparison with the analogous result presented in [10] reveals that the velocities of plane and axisymmetric diffusor and jet flows with radial stream lines depend on the polar angle θ in exactly the same manner. While the complete Navier-Stokes equations yield a solution which represents a strictly radial diffusor or jet flow in the plane case (see the Jeffery - Hamel solution in [8 and 10]), no such integral exists in the axisymmetric case.

For exact solutions of the Navier-Stokes equations of the form $\psi = rF(\theta)$ see the papers of H. B. Squire [12 and 13].

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APPENDIX A

TABLE 1: Eigenvalues $\lambda = \varkappa_1 + i\varkappa_2$ of Axisymmetric Slow Motions Past Conical Surfaces ($12° \leq 2\alpha \leq 160°$)

$2\alpha = 12°$ $\varkappa_1$	$2\alpha = 16°$ $\varkappa_1$	$2\alpha = 20°$ $\varkappa_1$	$2\alpha = 24°$ $\varkappa_1$	$2\alpha = 28°$ $\varkappa_1$	$2\alpha = 32°$ $\varkappa_1$	$2\alpha = 36°$ $\varkappa_1$	$2\alpha = 40°$ $\varkappa_1$	$2\alpha = 60°$ $\varkappa_1$	$2\alpha = 90°$ $\varkappa_1$	$2\alpha = 120°$ $\varkappa_1$	$2\alpha = 150°$ $\varkappa_1$	$2\alpha = 160°$ $\varkappa_1$
.3333*	.3386*	.3455*	.3539*	.3639*	.3754*	.3886*	.4033*	.5003*	.7118*	1*	1.3972	1.5668
1.5162	1.5274	1.5420	1.5604	1.5830	1.6100	1.6424	1.6813	1.9406	2.1805	2.4957	2.9123	3.0832
1.9999	1.9997	1.9993	1.9984	1.9969	1.9944	1.9902	1.9832					
2.6229	2.6396	2.6623	2.6924	2.7322	2.7880	2.8985	2.9235	3.1072	3.5122	3.9926	4.6278	4.8844
2.9996	2.9988	2.9968	2.9927	2.9843	2.9655							
$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	.1525	.3064	.3632	.2958	.1747
0	0	0	0	0	0	0	0					
0	0	0	0	0	0	.0399	.1123	.2964	.4214	.4743	.4304	.3453
0	0	0	0	0	0							

* Indicates singular eigenvalues.

TABLE 2: Eigenvalues $\lambda = \varkappa_1 + i\varkappa_2$ of Axisymmetric Slow Motions Past Conical Surfaces ($164° \le 2\alpha \le 270°$)

$2\alpha=164°$ $\varkappa_1$	$2\alpha=168°$ $\varkappa_1$	$2\alpha=172°$ $\varkappa_1$	$2\alpha=176°$ $\varkappa_1$	$2\alpha=180°$ $\varkappa_1$	$2\alpha=184°$ $\varkappa_1$	$2\alpha=188°$ $\varkappa_1$	$2\alpha=192°$ $\varkappa_1$	$2\alpha=196°$ $\varkappa_1$	$2\alpha=200°$ $\varkappa_1$	$2\alpha=210°$ $\varkappa_1$	$2\alpha=240°$ $\varkappa_1$	$2\alpha=270°$ $\varkappa_1$
1.6419	1.7219	1.8076	1.8998	2	2.1101	2.2334	2.3769	2.5617				
									2.8170	3.0172	3.8408	5.2400
3.1024	3.0235	3.0057	3.0007	3	2.9993	2.9943	2.9778	2.9283				
3.2123	3.4475	3.6303	3.8109	4	4.2075	4.4544						
							4.8284	4.9516	5.0819	5.4408	6.8921	9.3278
		5.0321	5.0030	5	4.9970	4.9693						
4.9952	5.1113											
		5.4346	5.7217	6	6.3061	.	.	.	.	.	.	.
$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$	$\varkappa_2$
0	0	0	0	0	0	0	0	0				
									.1610	.4369	.9599	1.5689
0	0	0	0	0	0	0	0	0				
0	0	0	0	0	0	0						
							.1481	.3329	.4494	.6686	1.2194	1.9063
		0	0	0	0	0						
.2807	.1637											
		0	0	0	0	.	.	.	.	.	.	.

APPENDIX B

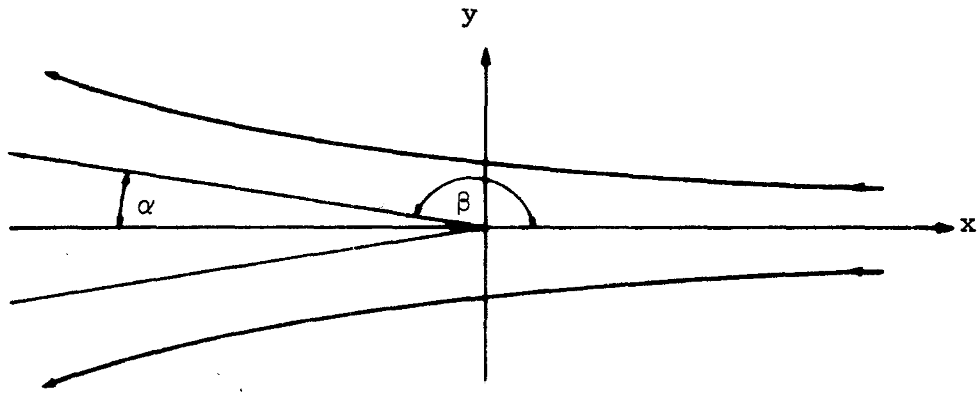

Fig. 1: Axisymmetric flow past a cone

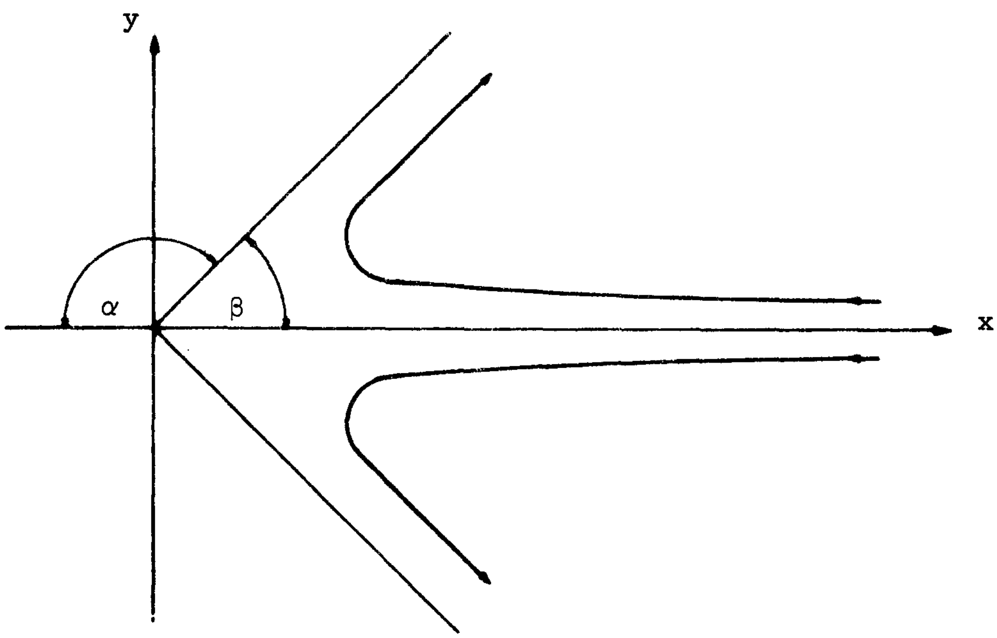

Fig. 2: Axisymmetric flow into a conical corner

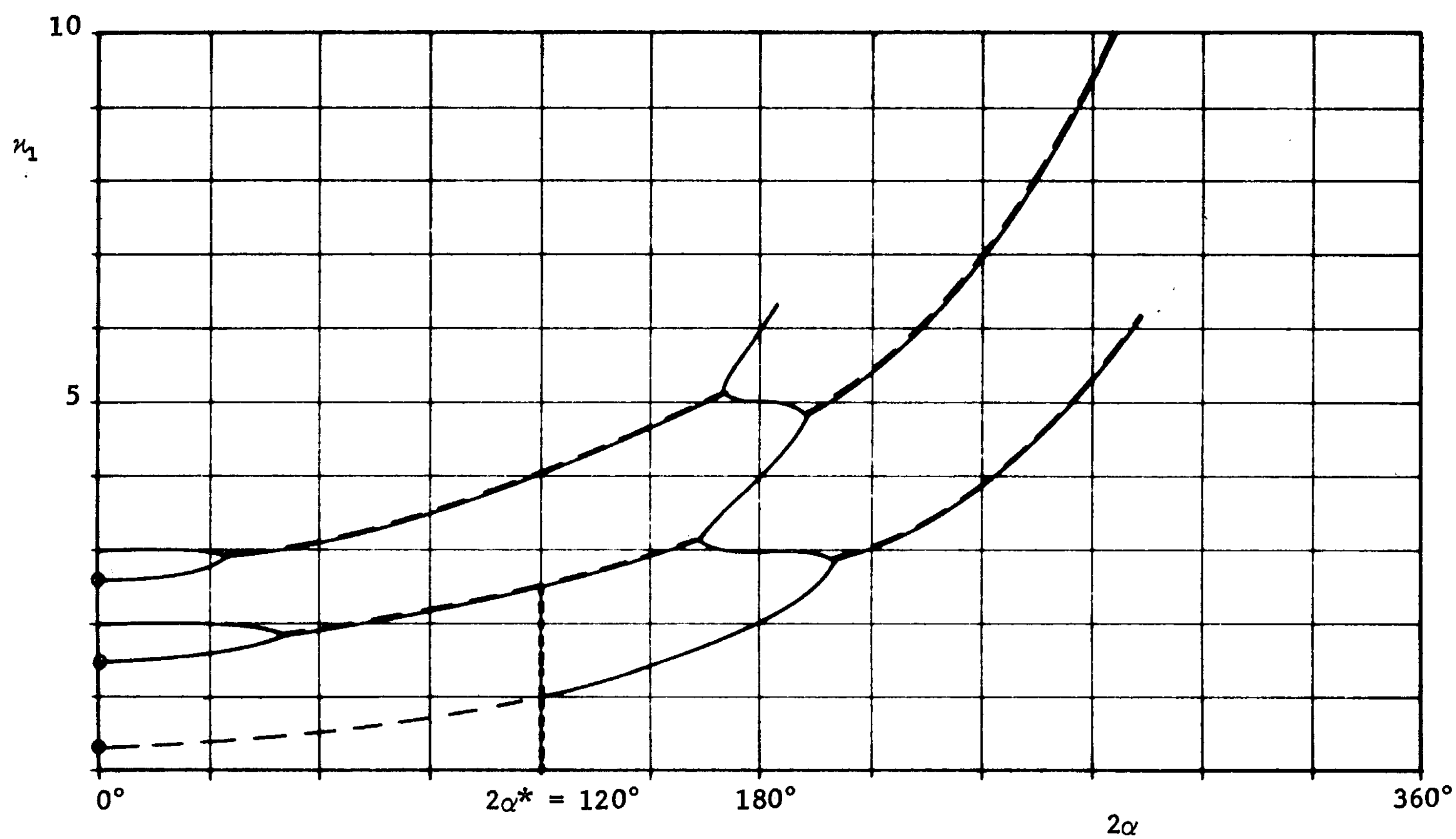

Fig. 3: Real parts of eigenvalues $\lambda = \varkappa_1 + i\varkappa_2$ of axisymmetric slow motions past a conical surface (2α = cone angle)

real eigenvalues — complex eigenvalues — singular eigenvalues a.

b.

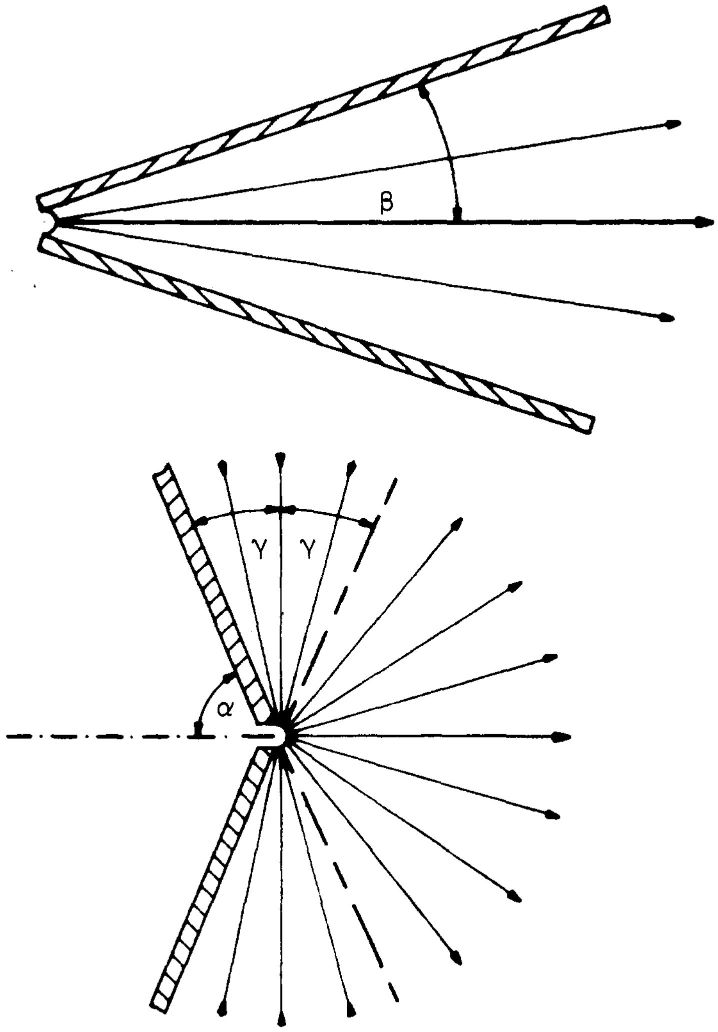

Fig. 4: a. Radial diffusor-type flow for $0 < 2\beta \leq \pi$.

b. Radial jet-type flow for $0 < 2\alpha < \pi$.

APPENDIX C

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Local:	
D	1
K	1
K-1	1
K-3	1
K-4	1
KXK	1
KXF	1
KXH	1
KXR	1
KYD	2
KXL	50
KYS	50
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BIBLIOGRAPHIC INFORMATION

DESCRIPTOR	CODE	DESCRIPTOR	CODE
SOURCE Naval Weapons Laboratory	NPGA	SECURITY CLASSIFICATION AND CODE COUNT UNCLASSIFIED	U016
REPORT NUMBER 1922	1922	CIRCULATION LIMITATION	
REPORT DATE May 1964	0564	CIRCULATION LIMITATION OR BIBLIOGRAPHIC	
		BIBLIOGRAPHIC *(Suppl., Vol., etc.)*	

SUBJECT ANALYSIS OF REPORT

DESCRIPTOR	CODE	DESCRIPTOR	CODE	DESCRIPTOR	CODE
Axisymmetrical	AXIA	Laminar	LAMI		
	SYMM	Flow	FLOW		
Viscosity	VISC				
Fluid	FLUI				
Motion	MOTI				
Conical	CONE				
Surfaces	SURA				
Partial	PARI				
Differential	DIFE				
Navier-Stokes	NAVE				
Equations	EQUA				
Eigenvalues	EIGV				
Problems	PRBL				
Mathematics	MATH				

Naval Weapons Laboratory (NWL Report No. 1922)
AXISYMMETRIC VISCOUS FLUID MOTIONS AROUND CONICAL SURFACES, by E. W. Schwiderski and others. May 1964. 22 p., 4 figs., 2 tables.

UNCLASSIFIED

Axisymmetric laminar motions around conical surfaces are investigated on the basis of complex Navier-Stokes equations. The solutions are represented in terms of complex slow-motion eigenfunctions. A note on diffusor-type and jet-type flows is added.

1. Viscous fluid motions
2. Partial differential equations
3. Eigenvalue problems
4. Mathematics
I. Schwiderski, E. W.
II. Lugt, H. J.
III. Ugincius, P.
IV. Title

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